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Design Sensitivity Analysis for Repeated Eigenvalues in Structural Design

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Eigenvalue sensitivity can be analytically computed using Fox and Kapoor's formula. However, if the eigenvalues are not distinct, the calculation of eigenvalue sensitivities for the repeated eigenvalues using Fox's formula sometimes does not yield correct answers unless the eigenvectors associated with the repeated eigenvalues are properly defined. It has been found in this paper that for two types of frequently designed structures, although the repeated eigenvalues exist, their sensitivities with respect to specific design variables can be found without forming an auxiliary eigenequation as proposed by earlier researchers. For one type of structure, the design variable has equal dynamic influence in two coordinate directions, and for the other type, the design variable affects the dynamic behavior in only one coordinate direction. Two numerical examples representing these two types of structures are given to demonstrate this finding.

Introduction

EIGENVALUE sensitivities are the derivatives of the eigenvalues with respect to some design variables that are available to the designers for structural modifications. The eigenvalue sensitivities provide very useful information to the designers for the redesign of the structures. Carefully using these calculated sensitivity data, the designers may modify the structures efficiently and also predict the eigenvalues of the modified structures. For these reasons, various methods for computing the first-order eigenvalue and the eigenvector sensitivities were introduced by Fox and Kapoori,[1] Rogers,[2] and Nelson,[3] and a survey was made by Adelman and Haftka.[4] Computing the eigenvalue sensitivities using an analytically derived formula involves the eigenvectors. For a system with distinct eigenvalues, the eigenvectors are uniquely defined. The previously mentioned methods can give the solutions without difficulty.

However, problems arise when eigenvalues of the structure are not distinct. The eigenvectors associated with those repeated eigenvalues are not unique. Instead, they can be any linear combination of the eigenvectors associated with the same repeated eigenvalue. Therefore the eigenvalue sensitivities for the repeated eigenvalues cannot be determined uniquely using those formulas.

In structural design, the repeated eigenvalues most often appear in those structures whose structural stiffness and mass are the same in two perpendicular coordinate directions, e.g., circular cross-sectional shafts, three-dimensional symmetric structures, etc. To find the eigenvalue sensitivities for repeated eigenvalues, Haug and Rousselet[5] and others[6,7] concluded that the repeated eigenvalue sensitivities were directional derivatives. The sensitivity data for these eigenvalues can be obtained by solving another eigensystem formed by the matrix product of the eigenvectors of the repeated eigenvalues as well as the derivatives of the stiffness and the mass matrices with respect to the specific design variable. Ojalvo[8] followed the previous researchers' works to compute the eigenvalue sensitivities for the repeated eigenvalues. He also extended Nelson's method to calculate the eigenvector derivatives for those eigenvectors that were associated with the repeated eigenvalues that had distinct eigenvalue sensitivities. Dailey[9] pointed out that Ojalvo's method to compute eigenvector derivatives only applied to some special cases. To correct this drawback, he suggested a method by taking the second derivative of the eigenequation with respect to the design parameter to find the weighting coefficients uniquely for the eigenvector sensitivity calculations. A more general approach thus resulted. Mills-Curran[10] found that the suggested approaches in Refs. 8 and 9 might fail to get a nonsingular matrix from the originally singular eigensystem matrix, and this nonsingular matrix was used to solve for the nonhomogeneous part of the eigenvector sensitivities. A simple example was illustrated to demonstrate his argument. Mills-Curran[11] further developed a more rigorous approach to find the sensitivities of those eigenvectors that were associated with repeated eigenvalues.

It has been found in this research that the aforementioned method of finding the eigenvalue sensitivities for repeated eigenvalues is not always necessary; i.e., under certain circumstances, there is no need to construct an auxiliary eigensystem to find the eigenvalue sensitivities for repeated eigenvalues. Explorations of this finding on two frequent design cases are discussed.

Theory

The analytical equation for computing eigenvalue sensitivity was developed by Fox and Kapoor[1] as follows:

$$\frac{\partial \lambda_i}{\partial \alpha} = \{\phi_i\}^T \left(\frac{\partial [K]}{\partial \alpha} - \lambda_i \frac{\partial [M]}{\partial \alpha} \right) \{\phi_i\} \tag{1}$$

where $\{\phi_i\}$ is the ith orthonormal eigenvector, α is the design variable, $[K]$ is the stiffness matrix, $[M]$ is the mass matrix of the structure, λ_i is the ith eigenvalue, and the superscript T represents the transpose of a matrix or a vector.

Owing to the nonunique nature of the eigenvectors that are associated with repeated eigenvalues, Eq. (1) in general does not produce a unique solution. To overcome this problem inherited from repeated eigenvalues, the following eigensystem is formed and solved[8]:

$$\left\{ [\Phi]^T \left(\frac{\partial [K]}{\partial \alpha} - \lambda_i \frac{\partial [M]}{\partial \alpha} \right) [\Phi] - \frac{\partial \lambda_i}{\partial \alpha} [I] \right\} \{a_i\} = \{0\} \tag{2}$$

where $[\Phi]$ contains the initially obtained orthonormal eigenvectors that are associated with the same repeated eigenvalue λ_i, and $[I]$ is the unit matrix with a dimension corresponding to the multiplicity of the repeated eigenvalue.

Equation (2) apparently is an eigensystem with $\partial \lambda_i / \partial \alpha$, the repeated eigenvalue sensitivity, being its eigenvalue and $\{a_i\}$

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the corresponding eigenvector. If the eigenvalues of this smaller eigensystem are distinct, then the unique eigenvector $\{a_i\}$ can be used to determine the unique eigenvector that should be associated with the repeated eigenvalues by the following linear combination of eigenvectors:

$$\{\bar{\phi}_i\} = [\Phi]\{a_i\} \tag{3}$$

where $\{\bar{\phi}_i\}$ stands for the unique eigenvector that corresponds to the repeated eigenvalue λ_i.

After determining the unique eigenvectors for the repeated eigenvalues, Eq. (1) can be used again to calculate the eigenvalue sensitivities for those repeated eigenvalues, although the solutions have already been found by solving the eigensystem of Eq. (2).

In practical structural designs, the symmetries of geometry, mechanical properties, and boundary conditions are sometimes unavoidable or favored. This type of structure has a very high chance to generate several sets of double eigenvalues, e.g., a supported rotating shaft, a square symmetric structure with symmetric supports, a cantilever beam with the same stiffness in two flexural directions, etc. The discussions of eigenvalue sensitivities for this type of structure will be divided into two cases. The first case deals with the design variable whose change affects the structural dynamic behavior equally in two coordinate directions. The second case deals with the design variable whose change affects the structural dynamic behavior in only one direction.

Case 1: Global Design Variable

For this type of design variable, the equal sensitivity for the repeated eigenvalues is expected since the change of the design variable gives the same influence in two perpendicular directions. The eigenvalue sensitivity for the repeated eigenvalues can be found using Eq. (1). There is no need to form Eq. (2) to calculate the eigenvalue sensitivities for this type of structure. The reason is shown next.

Assume

$$\{\bar{\phi}_1\} = [\Phi]\{c_1\} \tag{4}$$

and

$$\{\bar{\phi}_2\} = [\Phi]\{c_2\} \tag{5}$$

where $\{\bar{\phi}_1\}$and $\{\bar{\phi}_2\}$ represent two linearly combined eigenvectors associated with the repeated eigenvalues. Also the two original eigenvectors in $[\Phi]$ are assumed to be mutually orthogonal.

To normalize $\{\bar{\phi}_1\}$ and $\{\bar{\phi}_2\}$ such that $\{\bar{\phi}_1\}^T[M]\{\bar{\phi}_1\} = \{\bar{\phi}_2\}^T[M]\{\bar{\phi}_2\} = 1$, constraints are added to the elements of $\{c_1\}$ and $\{c_2\}$ as follows, where $i = 1, 2$,

$$\begin{aligned}\{\bar{\phi}_1\}^T[M]\{\bar{\phi}_i\} &= \{c_i\}^T[\Phi]^T[M][\Phi]\{c_i\}\\ &= \{c_i\}^T[I]\{c_i\}\\ &= c_{1i}^2 + c_{2i}^2\\ &= 1\end{aligned} \tag{6}$$

where c_{1i} and c_{2i} are the two elements of $\{c_i\}$, and $[I]$ is the unit matrix.

The constraint for normalization thus requires that

$$c_{1i} = \cos\theta_i, \qquad i = 1, 2 \tag{7}$$

$$c_{2i} = \sin\theta_i, \qquad i = 1, 2 \tag{8}$$

where θ_i is an arbitrary angle.

To maintain orthogonality between $\{\bar{\phi}_1\}$, and $\{\bar{\phi}_2\}$, another constraint must be put on $\{c_i\}$. That is,

$$\begin{aligned}\{\bar{\phi}_1\}^T[M]\{\bar{\phi}_2\} &= \{c_1\}^T[\Phi]^T[M][\Phi]\{c_2\}\\ &= \{c_1\}^T[I]\{c_2\}\\ &= \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\\ &= 0\end{aligned} \tag{9}$$

Equation (9) can be easily reorganized as

$$\cot\theta_1 = -\tan\theta_2 \tag{10}$$

The solution for Eq. (10) is

$$\theta_2 = \theta_1 + 90 \text{ deg} \tag{11}$$

It is clear that there are indeed an infinite number of eigenvectors that are qualified to be the eigenvectors of the repeated eigenvalues as long as Eqs. (7), (8), and (11) are satisfied. However, substituting any of these eigenvectors into Eq. (1) to compute the eigenvalue sensitivity for the repeated eigenvalues still results in a unique solution. This fact is explained as follows, where $i = 1, 2$,

$$\begin{aligned}\frac{\partial\lambda_i}{\partial\alpha} &= \{\bar{\phi}_i\}^T\left(\frac{\partial[K]}{\partial\alpha} - \lambda_i\frac{\partial[M]}{\partial\alpha}\right)\{\bar{\phi}_i\}\\ &= \{c_i\}^T[\Phi]^T\left(\frac{\partial[K]}{\partial\alpha} - \lambda_i\frac{\partial[M]}{\partial\alpha}\right)[\Phi]\{c_i\}\\ &= \{c_i\}^T\begin{bmatrix} X & Z\\ Z & Y\end{bmatrix}\{c_i\}\\ &= X\cos^2\theta_i + Y\sin^2\theta_i + 2Z\cos\theta_i\sin\theta_i\end{aligned} \tag{12}$$

where

$$\begin{bmatrix} X & Z\\ Z & Y\end{bmatrix} = [\Phi]^T\left(\frac{\partial[K]}{\partial\alpha} - \lambda_i\frac{\partial[M]}{\partial\alpha}\right)[\Phi] \tag{13}$$

Because the design variable α is assumed to have equal dynamic influence in two coordinate directions, the eigenvalue sensitivities for the double eigenvalues have to be the same. Based on this observation, the only chance to have equal eigenvalue sensitivity using Eq. (12) with arbitrary θ_1 and θ_2 is $Z = 0$ and $X = Y$. Since X and Y are the eigenvalue sensitivities using Eq. (1) with original eigenvectors, the fact that the eigenvalue sensitivities for the repeated eigenvalues can still be obtained using Fox's formula without any treatment of the original eigenvectors is proved.

Case 2: Directional Design Variable

In this case, although repeated eigenvalues exist in the initial design, after modification the eigenvalues become distinct. Therefore the eigenvalue sensitivities for the double eigenvalues are expected to be different. Furthermore, if the dynamic influence of the design variable is completely limited in one coordinate direction, then one of the eigenvalue sensitivities for the repeated eigenvalues is zero. Examples will be given later. Before using Eq. (2) to find the eigenvalue sensitivities, care has to be taken to insure that the eigenvectors associated with repeated eigenvalues are orthogonal to each other (since it has been found that some commercially available eigensolvers sometimes do not yield mutually orthogonal eigenvectors that are associated with repeated eigenvalues). If the eigenvectors associated with the repeated eigenvalues are not mutually orthogonal, Eq. (2) has to be modified in the following form:

$$\left\{[\Phi]^T\left(\frac{\partial[K]}{\partial\alpha}-\lambda_i\frac{\partial[M]}{\partial\alpha}\right)[\Phi]-\frac{\partial\lambda_i}{\partial\alpha}[\Phi]^T[M][\Phi]\right\}\{a_i\}=\{0\} \tag{14}$$

The eigenvalue sensitivities for the repeated eigenvalues can be obtained by solving the eigenproblem of Eq. (14), and then the unique eigenvector associated with the repeated eigenvalue is determined by Eq. (3). Further examination of Eq. (14) reveals a clearer picture of the matter. Equation (14) can be expressed as

$$\left\{\begin{bmatrix} X & Z \\ Z & Y \end{bmatrix}-\frac{\partial\lambda_i}{\partial\alpha}\begin{bmatrix} 1 & W \\ W & 1 \end{bmatrix}\right\}\{a_i\}=\{0\} \tag{15}$$

where W, X, Y, and Z are the corresponding elements in resultant matrices of those matrix products in Eq. (14). The eigenvalues of Eq. (15) have a closed-form solution as follows:

$$\frac{\partial\lambda_i}{\partial\alpha}=\frac{-2ZW+X+Y\pm\sqrt{(X-Y)^2+4Z^2-4W(ZX+ZY-XYW)}}{2(1-W)^2} \tag{16}$$

If one of the eigenvalue sensitivities is going to be zero, a simple relation between X, Y, and Z holds. Assuming $|W|\neq 1$ and letting the numerator of Eq. (16) be zero yields

$$-2ZW+X+Y=\mp\sqrt{(X-Y)^2+4Z^2-4W(ZX+ZY-XYW)} \tag{17}$$

Taking squares on both sides of Eq. (17) and simplifying the expressions result in a very simple relation among W, X, Y, and Z as follows:

$$(W^2-1)XY=(W^2-1)Z^2 \tag{18}$$

Since the mass matrix $[M]$ is always positive definite and the two eigenvectors are different, $|W|$ cannot be equal to 1. Equation (18) is further simplified to be

$$XY=Z^2 \tag{19}$$

This relation is very reasonable. The first matrix in Eq. (15) is singular if Eq. (19) is valid. To make the resultant matrix in the parentheses singular, one apparent choice is to let $\partial\lambda_i/\partial\alpha$ be zero. If the two eigenvectors associated with the repeated eigenvalues are orthogonal to each other, then W equals zero. Under this condition, Eq. (16) becomes

$$\frac{\partial\lambda_i}{\partial\alpha}=\frac{X+Y\pm\sqrt{(X-Y)^2+4Z^2}}{2} \tag{20}$$

After one substitutes XY for Z^2 in Eq. (20), the eigenvalue sensitivities are

$$\begin{aligned}\frac{\partial\lambda_i}{\partial\alpha}&=\frac{X+Y\pm(X+Y)}{2}\\ &=0 \qquad \text{or} \qquad X+Y\end{aligned} \tag{21}$$

For the nonzero eigenvalue sensitivity, the actual mathematical expression is

$$\begin{aligned}\frac{\partial\lambda_i}{\partial\alpha}&=X+Y\\ &=\sum_{r=1}^{2}\{\phi_r\}^T\left(\frac{\partial[K]}{\partial\alpha}-\lambda_i\frac{\partial[M]}{\partial\alpha}\right)\{\phi_r\}\end{aligned} \tag{22}$$

where $\{\phi_r\}$ is the original orthonormal eigenvector associated with the repeated eigenvalue λ_i.

It is obvious from Eq. (22) that if one of the eigenvalue sensitivities is zero for the double eigenvalues, the other nonzero eigenvalue sensitivity equals the sum of Eq. (1) for the two orthonormal eigenvectors. Therefore there is again no need to solve Eq. (2) to find the eigenvalue sensitivities for repeated eigenvalues.

In case 2, since the eigenvalue sensitivities are different for the repeated eigenvalues, the eigenvector sensitivities exist. Using Mills-Curran's[11] approach to compute the eigenvector sensitivities associated with the repeated eigenvalues, the eigenvectors were assumed to be mutually orthogonal. If the eigenvectors recovered by Eq. (3) are not mutually orthogonal, they can be modified as follows.

Let

$$\{\phi_i^0\}=[\bar\Phi]\{b_i\}, \qquad i=1,2 \tag{23}$$

where $\{\phi_i^0\}$ represents the mutually orthogonal eigenvector for the double eigenvalues, $[\bar\Phi]$ contains the previously adjusted eigenvectors using Eq. (3) and $\{b_i\}$ is an unknown vector to be determined. To be an orthonormal set of eigenvectors, the following equation has to be satisfied:

$$[\Phi^0]^T[M][\Phi^0]=[I] \tag{24}$$

where $[\Phi^0]$ is composed of the two eigenvectors of Eq. (23).

Substituting Eq. (23) into Eq. (24) yields

$$\begin{aligned}&[\{b_1\}\{b_2\}]^T[\bar\Phi]^T[M][\bar\Phi][\{b_1\}\{b_2\}]\\ &=\begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22}\end{bmatrix}\begin{bmatrix} 1 & V \\ V & 1\end{bmatrix}\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22}\end{bmatrix}\\ &=\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\end{aligned} \tag{25}$$

where b_{ij} represents the ith element of the jth vector and

$$\begin{bmatrix} 1 & V \\ V & 1\end{bmatrix}=[\bar\Phi]^T[M][\bar\Phi] \tag{26}$$

$$V=\{\bar\phi_1\}^T[M]\{\bar\phi_2\} \tag{27}$$

Three algebraic equations are extracted from Eq. (25):

$$b_{11}^2+b_{21}^2+2Vb_{11}b_{21}=1 \tag{28}$$

$$b_{12}^2+b_{22}^2+2Vb_{12}b_{22}=1 \tag{29}$$

$$b_{11}b_{12}+b_{21}b_{22}+Vb_{21}b_{12}+Vb_{11}b_{22}=0 \tag{30}$$

Solving these nonlinear equations gives

$$b_{11}=1 \tag{31}$$

$$b_{21}=0 \tag{32}$$

$$b_{12}=-Vb_{22} \tag{33}$$

$$b_{22}=\sqrt{\frac{1}{1-V^2}} \tag{34}$$

Table 1 Eigenvalues of examples 1 and 2

Mode no.	Example 1	Example 2
1	452	495
2	452	495
3	9,948	11,977
4	9,948	11,977
5	48,492	93,333
6	48,492	93,333
7	107,593	200,859
8	107,593	200,859

Table 2 Eigenvalue sensitivities for example 1

Mode no.	$\partial\lambda/\partial\alpha$, Eq. (1)	$\partial\lambda/\partial\alpha$, Eq. (2)	$\partial\lambda/\partial\alpha$, finite difference	X	Y	Z
1	38.8	38.8	38.8	38.8	38.8	0
2	38.8	38.8	38.8	38.8	38.8	0
3	228.4	228.4	228.3	228.4	228.4	0
4	228.4	228.4	228.3	228.4	228.4	0
5	2006.6	2006.6	2007.5	2006.6	2006.6	0
6	2006.6	2006.6	2007.5	2006.6	2006.6	0
7	−449.8	−449.8	−450	−449.8	−449.8	0
8	−449.8	−449.8	−450	−449.8	−449.8	0

Table 3 Eigenvalue sensitivities for example 2

Mode no.	$\partial\lambda/\partial\alpha$, Eq. (22)	$\partial\lambda/\partial\alpha$, Eq. (2)	$\partial\lambda/\partial\alpha$, finite difference	X	Y	Z	XY	Z^2
1	0	0	0	1.586	11.973	−4.358	18.99	18.99
2	13.56	13.56	13.56	1.586	11.973	−4.358	18.99	18.99
3	0	0	0	19.25	145.33	−52.89	2,797.8	2,797.8
4	164.6	164.6	165	19.25	145.33	−52.89	2,797.8	2,797.8
5	0	0	0	169.18	1277.06	−464.81	216,052	216,052
6	1446.2	1446.2	1446	169.18	1277.06	−464.81	216,052	216,052
7	0	0	0	79.03	596.58	−217.14	47,149.4	47,149.4
8	675.6	675.6	675	79.03	596.58	−217.14	47,149.4	47,149.4

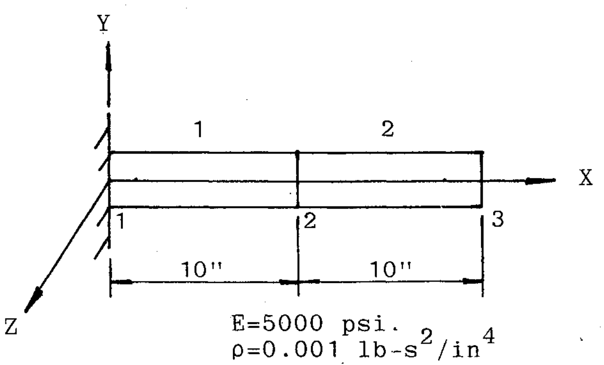

Fig. 1 Cantilever beam.

According to the solution, only one eigenvector needs to be further modified to have orthogonal eigenvectors for the repeated eigenvalues.

Numerical Examples

The first example is a two-element cantilever beam with uniform circular cross-sectional area of 20 $in.^2$. Two transverse translational degrees of freedom and two corresponding rotational degrees of freedom are allowed at each free node. The dimension and the material properties are shown in Fig. 1. The lumped mass matrix is used in finite element analysis. The rotational mass moments of inertias are 1.985 and 0.992 in.-lb-s^2 at nodes 2 and 3, respectively. The design variable is chosen to be the cross-sectional area of the first element. The variations of the bending moments of inertias in the y and z directions are the same with respect to this design variable. Therefore, this example is used to demonstrate the case 1 structures.

The second example has a structure that is similar to that of the first example except that the cross-sectional area is square. The bending moments of inertias in the y and z directions are the same in the initial design. The design variable is designated to be I_{zz} of the first element. The rotational mass moments of inertias are 1.0 and 0.5 in.-lb-s^2 at nodes 2 and 3, respectively. It is obvious that the variation of I_{zz} affects bending behavior in the y direction only. For any pair of repeated bending modes, one of the eigenvalue sensitivities would be zero. Therefore, this example is used to illustrate the case 2 structures.

The eigenvalues of the two example structures are shown in Table 1. Four pairs of repeated bending modes appear. The eigenvalue sensitivities for these two examples using different formulations are given in Tables 2 and 3, respectively.

Conclusions

The calculations of sensitivities for the repeated eigenvalues were proposed by previous researchers to form an auxiliary eigenproblem using eigenvectors associated with repeated eigenvalues. The eigenvalues of this smaller eigenproblem are the eigenvalue sensitivities of the repeated eigenvalues in the original system. It is found in this research that for two special cases, there is no need to follow the previously suggested procedures to solve another eigenproblem for the eigenvalue sensitivities of the repeated roots. The eigenvalue sensitivities for the repeated roots can still be obtained using the Fox formula provided the design variable has global influence on the structural matrices. On the other hand, if the design variable has only local influence on the structural matrices, then one of the eigenvalue sensitivities must be zero and the other one is the sum of the results by the Fox formula using the two associated eigenvectors. Two simple examples demonstrate these findings.

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